Scaling in Tournaments

E. Ben-Naim,[1] S. Redner,[2] and F. Vazquez[3, 2]

[1] *Theoretical Division and Center for Nonlinear Studies, Los Alamos National Laboratory, Los Alamos, New Mexico 87545 USA*

[2] *Department of Physics, Boston University, Boston, Massachusetts 02215 USA*

[3] *Theoretical Division and Center for Nonlinear Studies, Los Alamos National Laboratory, Los Alamos, New Mexico 87545*

We study a stochastic process that mimics single-game elimination tournaments. In our model, the outcome of each match is stochastic: the weaker player wins with upset probability $q \le 1/2$, and the stronger player wins with probability $1-q$. The loser is then eliminated. Extremal statistics of the initial distribution of player strengths governs the tournament outcome. For a uniform initial distribution of strengths, the rank of the winner, x_*, decays algebraically with the number of players, N, as $x_* \sim N^{-\beta}$. Different decay exponents are found analytically for sequential dynamics, $\beta_{\rm seq} = 1-2q$ and parallel dynamics, $\beta_{\rm par} = 1 + \frac{\ln(1-q)}{\ln 2}$. The distribution of player strengths becomes self-similar in the long time limit and moreover, it has an algebraic tail.

PACS numbers: 01.50.Rt, 02.50.-r, 05.40.-a, 89.75.Da

A wide variety of processes in nature and society involve competition. In animal societies, competition is responsible for social differentiation and the emergence of social strata. Competition is also ubiquitous in human society: auctions, election of public officials, city plans, grant awards, and sports are often decided by competition. Minimalist, physics-based competition processes have been recently developed to model relevant competitive phenomena such as wealth distributions [1–3], auctions [4–6], social dynamics [8–10], and sports leagues [11]. In physics, competition also underlies phase ordering kinetics, in which large domains grow at the expense of small domains that eventually are eliminated [12, 13].

In this study, we investigate N-player tournaments with head-to-head matches. The winner of each match remains in the tournament while the loser is eliminated. At the end of a tournament, a single undefeated player, the tournament winner, remains. Each player is endowed with a fixed intrinsic strength $x \ge 0$ that is drawn from a normalized distribution $f_0(x)$. For convenience, we define strength so that smaller x corresponds to a stronger player and we henceforth refer to this strength measure as "rank".

The result of competition is stochastic: in each match the weaker player wins with probability $q \le 1/2$ and the stronger player wins with probability $p = 1-q$. Schematically, when two players with ranks x_1 and x_2 compete, assuming $x_1 > x_2$, the outcome is:

$$(x_1, x_2) \to \begin{cases} x_1 & \text{with probability } 1-q; \\ x_2 & \text{with probability } q. \end{cases} \tag{1}$$

For $q = 0$, the best player is always victorious, while for $q = 1/2$, outcomes are completely random. We restrict our considerations to the case $q \le 1/2$, as the complementary situation follows by symmetry arguments. We are interested in the evolution of the rank distribution, as well as the rank of the tournament winner.

We find that the rank of the winner, x_*, decays algebraically with the number of players N as

$$x_* \sim N^{-\beta} \tag{2}$$

with the exponent $\beta \equiv \beta(q)$ a function of the upset probability. We emphasize that Eq. (2) implies that the strongest player does not necessarily win the tournament [14]. When the ranks of the tournament players are uniformly distributed, we find different values for sequential and parallel dynamics: $\beta_{\rm seq} = 1-2q$ and $\beta_{\rm par} = 1 + \frac{\ln(1-q)}{\ln 2}$. Moreover, the rank distribution becomes asymptotically self-similar and it has a power-law tail. We also extend these results to arbitrary initial distributions. Generally, extremal properties of the initial distribution of player strength govern statistical properties of the rank of the ultimate winner.

We formulate the competition process dynamically by assuming that each pair of players compete at a constant rate. In this formulation, games are held sequentially, and players are eliminated from the tournament one at a time. Then the fraction of players remaining in the competition at time t, $c(t)$, decays according to

$$\frac{dc}{dt} = -c^2. \tag{3}$$

Solving this equation subject to the initial condition, $c(0) = 1$, the surviving fraction is

$$c(t) = (1+t)^{-1}. \tag{4}$$

The tournament ends with a single player and this occurs at time t_*, that can be estimated from $c(t_*) = N^{-1}$. Therefore the time to complete the competition scales linearly with the number of players $t_* \simeq N$. There are other ways to define the dynamics; for example, the number of matches played is an equally sensible measure of time. Our choice leads to slightly simpler expressions, but as long as the two competitors are chosen randomly, these two ways for characterizing time are equivalent.

We focus on the evolution of the fraction of players with a given rank. Let $f(x,t)\,dx$ be the fraction of players with rank in the range $(x, x+dx)$ at time t. This density obeys the nonlinear integro-differential equation

$$\frac{\partial f(x)}{\partial t} = -2p\, f(x) \int_0^x dy f(y) - 2q\, f(x) \int_x^\infty dy f(y). \quad (5)$$

The first term accounts for games where the favorite wins and the second term for games where the underdog wins. The initial condition is $f(x,0) = f_0(x)$ with $\int dx f_0(x) = 1$.

The rank distribution can be obtained by introducing the cumulative distribution $F(x)$, defined as the fraction of players with rank smaller than x,

$$F(x) = \int_0^x dy f(y). \quad (6)$$

The distribution of player ranks is obtained from the cumulative distribution by differentiation, $f(x) = dF(x)/dx$, while the total fraction of players remaining in the competition is simply $c(t) = \lim_{x\to\infty} F(x,t)$. By integrating the master equation (5), the cumulative distribution obeys the closed nonlinear ordinary differential equation

$$\frac{\partial F}{\partial t} = (2q-1)F^2 - 2qnF. \quad (7)$$

The initial condition is $F(x,0) = F_0(x) = \int_0^x dy f_0(y)$. We solve this equation using the transformation $G(x) = 1/F(x)$ [15] to reduce (7) to the linear equation

$$\frac{\partial G}{\partial t} = (1-2q) + 2qnG. \quad (8)$$

Integrating this equation with respect to time, we find $G(x) = [G_0(x) - 1](1+t)^{2q} + (1+t)$. Substituting the initial condition $G_0(x) = 1/F_0(x)$, we arrive at our first result, the cumulative distribution of rank as a function of time:

$$F(x,t) = \frac{F_0(x)}{[1-F_0(x)](1+t)^{2q} + F_0(x)(1+t)}. \quad (9)$$

From this, the actual density of player rank is obtained by differentiation

$$f(x,t) = \frac{f_0(x)(1+t)^{2q}}{\left[(1-F_0(x))(1+t)^{2q} + F_0(x)(1+t)\right]^2}. \quad (10)$$

When the game outcome is random, $q = 1/2$, then the properly normalized distribution of rank does not evolve with time as $f(x,t)/c(t) = f_0(x)$.

Uniform Initial Distribution. To illustrate some general features, consider the special case of a uniform initial distribution, $f_0(x) = 1$ for $0 \le x \le 1$, and deterministic games, $q = 0$. Then the initial cumulative distribution is $F_0(x) = x$ for $x \le 1$ and $F_0(x) = 1$ for $x \ge 1$. The time-dependent cumulative distribution (9) is

$$F(x,t) = \frac{x}{1+xt} \quad (11)$$

for $x \le 1$ and $F(x,t) = c(t)$ for $x \ge 1$. Similarly, the rank distribution itself is $f(x,t) = (1+xt)^{-2}$ for $0 \le x \le 1$. Notice that for increasing rank x (weaker player), there are fewer players with this rank. Thus surviving players are progressively stronger as the tournament proceeds. Quantitatively, the average rank of surviving players, $\langle x\rangle = \int dx\, x f(x)/\int dx f(x)$, is

$$\langle x\rangle = t^{-2}\left[(1+t)\ln(1+t) - t\right]. \quad (12)$$

Therefore, the average rank asymptotically decays with time, $\langle x\rangle \simeq t^{-1}\ln t$, indicating that better players survive to the late stages of the tournament.

We can write the cumulative distribution in the scaling form $F(x,t) \to t^{-1}\Phi(xt)$, by multiplying and dividing (11) by time. Here, the scaling function is $\Phi(z) = \frac{z}{1+z}$, which approaches unity $\Phi(z) \to 1$ when $z \to \infty$, consistent with total density decay $c \simeq t^{-1}$. In the long time limit, the cumulative distribution retains the same shape as the initial distribution, $\Phi(z) \simeq z$, for $z \ll 1$. The scaling variable $z = xt$ indicates that players with rank larger than the characteristic rank $x \sim t^{-1}$ are eliminated from the tournament.

Let us generalize these results to arbitrary q. In this case, the cumulative distribution is

$$F(x,t) = \frac{x}{(1-x)(1+t)^{2q} + x(1+t)}, \quad (13)$$

for $x \le 1$ and $F(x,t) = c(t)$ otherwise. In the long time limit, we may replace $1+t$ with t, and also replace $1-x$ with 1, since the rank decays with time. Then the cumulative distribution approaches the scaling form

$$F(x,t) \to t^{-1}\Phi\left(x\, t^{1-2q}\right). \quad (14)$$

The scaling function remains as above

$$\Phi(z) = \frac{z}{1+z}. \quad (15)$$

The scaling form (14) implies that the typical rank decays algebraically with time

$$x \sim t^{-(1-2q)}. \quad (16)$$

Interestingly, the exponent governing this decay depends on the upset probability. The larger the upset probability, the smaller the decay exponent. Thus weaker players can persist in a tournament when q approaches 1/2. For completely random games, $q = 1/2$, the exponent vanishes and the strength of the typical surviving player does not change with time.

A similar scaling law characterizes the rank of the tournament winner as a function of the number of players. From (4), the number of players remaining in the tournament, M, and the initial number of players N, are related by $t \sim N/M$. Using (16), when M players remain, the typical rank is $x \sim (N/M)^{-(1-2q)}$. Substituting $M = 1$, we arrive at our second main result: the typical rank of the winner decays algebraically with the total number of players, as in (2), with the exponent

$$\beta_{\rm seq} = 1 - 2q. \tag{17}$$

Therefore, the smaller the tournament or the higher the upset probability the weaker the winner, on average. Intuitively, small tournaments require only a small number of upsets to produce a surprise winner.

General Initial Distributions. The findings in the case of uniform distributions suggest that the behavior of the initial distribution in the $x \to 0$ limit governs the long time asymptotics. Let us consider rank distributions with a power-law behavior near the origin,

$$F_0(x) \simeq C\, x^{\mu+1}, \tag{18}$$

as $x \to 0$ with $\mu > -1$ so that the distribution is normalized. The rank density then scales as $f_0(x) \simeq C(\mu+1)x^\mu$. Since the rank x decays with time, the term $(1-F_0)(1+t)^{2q}$ in the denominator can be replaced by t^{2q} and similarly, the term $F_0(x)(1+t)$ can be replaced by $Cx^{\mu+1}t$. Therefore, the scaling form (14) becomes $F(x,t) \to t^{-1}\Phi\left(x\, t^{\frac{1-2q}{\mu+1}}\right)$, with the scaling function $\Phi(z) = Cz^{\mu+1}/(1+Cz^{\mu+1})$. Thus the typical player rank decays with time according to $x \sim t^{-\frac{1-2q}{\mu+1}}$. Similarly, the rank of the winner decays with the number of players as in (2) with $\beta_{\rm seq} = \frac{1-2q}{\mu+1}$.

We conclude that extreme properties of the initial distribution fully governs the asymptotic behavior. In the long time limit, the player distribution becomes self-similar. Both the form of the scaling distribution and the time dependence of the characteristic rank depend only on the small-x behavior of the initial distribution. As the tournament progresses, weaker players are gradually eliminated and the initial distribution of player strength governs the asymptotic rank distribution. A similar phenomenology where extremal statistics governs long-time asymptotics was found in studies of clustering in traffic flows [16] and species abundance in biological evolution [17, 18].

Like the cumulative distribution, the density of players with given rank also becomes self-similar asymptotically, $f(x,t) \to t^{\beta-1}\phi\big(x\, t^\beta\big)$ with $\beta = \frac{1-2q}{\mu+1}$ and $\phi(z) = \Phi'(z)$. As noted earlier, the shape of the distribution is preserved $f(z) \sim z^\mu$ as $z \to 0$. The large argument behavior is

$$\phi(z) \sim z^{-\mu-2}, \tag{19}$$

as $z \to \infty$. This algebraic decay shows that the likelihood of finding weak players in the tournament is appreciable.

The scaling behavior (2) refers to the typical rank of the winner. The algebraic tail (19) suggests that the average rank may scale differently than the typical rank. For example, for compact uniform distributions ($\mu = 0$), the average is characterized by a logarithmic correction as in (12), $\langle x_* \rangle \sim N^{-(1-2q)} \ln N$.

In the limit $\mu \to \infty$, the typical rank decays slower than a power-law. When the distribution is sharply suppressed near the origin, $F_0(x) \sim \exp(-x^{-\nu})$, then the typical rank decays logarithmically with time, $x \sim (\ln t)^{-1/\nu}$, and correspondingly, the rank of the tournament winner scales as a function of the number of players as $x_* \sim (\ln N)^{-1/\nu}$.

Discrete Distributions. In many sports, the ranks of tournament players are discrete and it is straightforward to generalize our results to discrete rankings. Define f_k as the density of players of rank k, with $k = 1, 2, \dots$ and $F_k = \sum_{j=1}^{k} f_j$ as the corresponding cumulative distribution. Then the time-dependent distribution is a direct generalization of (9):

$$F_k(t) = \frac{F_k(0)}{[1-F_k(0)](1+t)^{2q} + F_k(0)(1+t)}. \tag{20}$$

In particular, the total density $c(t) = \lim_{k\to\infty} F_k$ is again $(1+t)^{-1}$, as in (4), while the rank density is obtained from $f_k = F_k - F_{k-1}$.

In spite of the equivalence for continuum and discrete distributions, the latter has the feature that the fraction of surviving top-ranked players, $f_1 \equiv F_1$, decays with time as $f_1 \simeq t^{-1}$, while from (20), the fraction of all other players decays as $f_k \simeq A_k t^{-2(1-q)}$, with $A_k = f_k(0)/[F_k(0)F_{k-1}(0)]$ for $k > 1$. In summary,

$$f_k \sim \begin{cases} t^{-1} & k = 1 \\ t^{-2(1-q)} & k > 1. \end{cases} \tag{21}$$

Parallel Dynamics. Thus far, we addressed games that are held sequentially with a single team eliminated at a time. However, actual sports tournaments typically proceed via round play in which games are held in parallel and half of the teams are eliminated in each round.

We thus consider such round-play tournaments with $N = 2^k$ players. Let $F_N(x)$ be the cumulative distribution of the rank of the tournament winner and let $f_N(x) = dF_N(x)/dx$ be the corresponding rank density. This distribution is now normalized $\int dx f_N(x) = 1$.

Consider first a tournament with $N = 2$ players. The rank distribution of the winner is

$$f_2(x) = 2pf_1(x)[1-F_1(x)] + 2qf_1(x)F_1(x). \tag{22}$$

By merely changing the sign, the right-hand-side becomes identical to the right-hand-side in Eq. (5). Integrating this equation, we arrive at an explicit expression for the distribution of the rank of the winner $F_2(x) = 2pF_1(x) + (1-2p)[F_1(x)]^2$. Clearly, this nonlinear recursion relation applies to every round of the tournament and therefore,

$$F_{2N}(x) = 2pF_N(x) + (1-2p)[F_N(x)]^2. \tag{23}$$

Iterating this equation starting with $F_1(x)$, we obtain explicit expressions for the distribution of the winner for $N = 2, 4, 8, \dots$ Explicit expressions can be obtained for the extreme cases of deterministic competitions ($q = 0$) where $1 - F_N(x) = [1 - F_1(x)]^N$ and random competitions

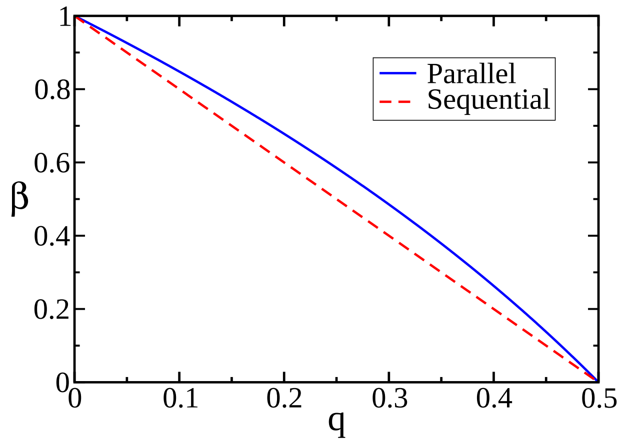

FIG. 1: The decay exponent β versus the upset probability q. Shown are the values for the sequential case (17) and the parallel case (26).

($q = 1/2$) where $F_N(x) = F_1(x)$. Generally, however, it is difficult to obtain the counterpart of the exact explicit solution (9), so we perform a scaling analysis.

Let us restrict our attention to uniform initial distributions as above, $F_1(x) = x$ for $x \le 1$. For small-x, we may neglect the nonlinear term in (23) and then, $F_2(x) \simeq (2p)\,x$, $F_4(x) \simeq (2p)^2\,x$, and in general

$$F_{2^k}(x) \simeq (2p)^k\,x. \tag{24}$$

To obtain the asymptotic behavior, we write $N = 2^k$. Substituting $k = \frac{\ln N}{\ln 2}$ into (24), then $F_N(x) \simeq N^\beta x$ with $\beta = 1 + \frac{\ln p}{\ln 2}$. Therefore, the cumulative distribution of the rank of the winner follows the scaling form

$$F_N(x) \to \Psi\left(xN^\beta\right) \tag{25}$$

when $N \to \infty$. The scaling function is linear, $\Psi(z) \simeq z$, in the limit $z \to 0$, reflecting that the extremal statistics are invariant under the competition dynamics.

The scaling form (25) shows that the rank of the tournament winner decays algebraically with the tournament size as in (2). Surprisingly, the decay exponent

$$\beta_{\rm par} = 1 + \frac{\ln(1-q)}{\ln 2} \tag{26}$$

for parallel dynamics, differs from the decay exponent (17) for sequential dynamics. The two exponents coincide in the extreme cases, $\beta(0) = 1$ and $\beta(1/2) = 0$. The inequality $\beta_{\rm par} \ge \beta_{\rm seq}$ shows that round play benefits the strong players (Figure 1). Indeed, in parallel play, all players must face competition in each and every round so that weak players are less likely to survive.

The source of the disparity between serial and parallel play is fluctuations in the number of games. In sequential dynamics, the number of games played by each player varies while in parallel dynamics the number of games is fixed. Typically, such fluctuations have a negligible effect in a broad class of stochastic processes. In elimination tournaments there are significant variations in the number of games played and this effect is strong enough to affect the scaling exponents.

Substituting the scaling form (25) into the recursion (23), the scaling function obeys the nonlinear and nonlocal equation

$$\Psi(2pz) = 2p\,\Psi(z) + (1-2p)\Psi^2(z). \tag{27}$$

The boundary condition are $\Psi(0) = 0$ and $\Psi(\infty) = 1$. An exact solution is feasible only when there are no upsets: $\Psi(z) = 1 - e^{-z}$ for $q = 0$. Otherwise, we perform an asymptotic analysis. As shown above, the small-z behavior is generic, $\Psi(z) \simeq z$. At large arguments, we write $U(z) = 1 - \Psi(z)$ and since $U \ll 1$, we can neglect the nonlinear terms and then $U(2pz) = 2qU(z)$. This implies the algebraic decay $U(z) \sim z^{1-\gamma}$ with $\gamma = 1 - \frac{\ln 2q}{\ln 2p}$. As a result, the likelihood of finding weak winners, $f_N(x) \to N^\beta \psi\left(xN^\beta\right)$ with $\psi(z) = \Psi'(z)$, decays algebraically

$$\psi(z) \sim z^{-1+\frac{\ln 2q}{\ln 2p}} \tag{28}$$

as $z \to \infty$. This algebraic behavior is very different from the exponential decay $\psi(z) = e^{-z}$ for deterministic games. In contrast with sequential play, the exponent depends on the upset probability. This large likelihood of finding weak winners reflects that the number of games played by the tournament winner scales logarithmically with the number of teams.

In summary, we studied the dynamics of single-elimination tournaments, in which there is a finite probability for a lower-ranked player to upset a higher-ranked player. We obtained an exact solution for the distribution of player ranks for arbitrary initial conditions. This distribution follows a scaling form in the long-time limit that is determined by the extremal properties of the initial distribution. Generally, the likelihood of upset winners is relatively large since the tail of the distribution function decays algebraically with rank. The characteristic rank of the winning player decays algebraically with the number of players and the larger the upset probability, the slower this decay. Different decay exponents are found for sequential and parallel play with the latter generally larger.

These results quantify three simple facts about tournament play. First, small tournaments are more likely to produce a surprise winner. Second, weak players fare better in sequential play where they may survive by being idle. Third, there is an appreciable probability for a tournament to produce a surprising outcome.

Our study was stated in terms of game dynamics. However, the underlying stochastic process is elementary. Starting with N independent and identically distributed random variables, elements are removed from the system sequentially. In each step, two variables are randomly picked and then with a fixed probability the larger is retained, while with the complementary probability, the smaller is retained. We anticipate that this simple model has relevance in other contexts. For example, this stochastic process also mimics the "survival of the fittest" in evolution.

It will be interesting to use this theory to model real tournament data, for example, by characterizing statistical properties of the rank of the winner as a function of the tournament size. Our theory provides a suite of predictions concerning the distribution of rank and the winning rank.

Acknowledgments. We thank the Isaac Newton Institute for Mathematical Sciences (Cambridge, England) and the Asia Pacific center for Theoretical Physics (Pohang, South Korea), where this research was largely performed, for their hospitality. We acknowledge financial support from DOE grant DE-AC52-06NA25396 and NSF grant DMR0535503.

[1] S. Ispolatov, P. L. Krapivsky, and S. Redner, Eur. Phys. Jour. B **2**, 267 (1998).
[2] A. Dragulescu and V. M. Yakovenko, Eur. Phys. Jour. B **17**, 723 (2000).
[3] F. Slanina, Phys. Rev. E **69**, 046102 (2004).
[4] R. D'Hulst, G. J. Rodgers, Physica A 294, 447 (2001).
[5] E. Ben-Naim and P. L. Krapivsky, Eur. Phys. Jour. B 25, 239 (2002).
[6] E. Ben-Naim, B. Kahng, J. S. Kim, J. Stat. Mech. P07001 (2006).
[7] E. Bonabeau, G. Theraulaz, and J.-L. Deneubourg, Physica A **217**, 373 (1995).
[8] E. Ben-Naim and S. Redner, J. Stat. Mech. L11002 (2005).
[9] K. Malarz, D. Stauffer, K. Kulakowski, Eur. Phys. Jour. B **50**, 195 (2006).
[10] E. Ben-Naim, F. Vazquez, and S. Redner, Eur. Phys. Jour. B **49**, 531 (2006).
[11] E. Ben-Naim, F. Vazquez, and S. Redner, physics/0512143, J. Quant. Anal. Sports, in press.
[12] J. D. Gunton, M. San Miguel, and P. S. Sahni in: *Phase Transitions and Critical Phenomena*, Vol. 8, eds. C. Domb and J. L. Lebowitz (Academic, NY 1983).
[13] A. J. Bray, Adv. Phys. **43**, 357 (1994).
[14] An intriguing example of this phenomenon for 3-way competition is presented in R. Toral and P. Amengual, cond-mat/0505388; P. Amengual and R. Toral, math.PR/0606181.
[15] C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers* (McGraw-Hill Book Co., Singapore, 1984).
[16] E. Ben-Naim, P.L. Krapivsky and S. Redner, Phys. Rev. E **50**, 822 (1994).
[17] K. Jain and J. Krug, J. Stat. Mech. P040068 (2005).
[18] C. Sire, S. N. Majumdar, D. S. Dean, cond-mat/0606101.